

# On the continuity of the commutative limit of the 4d $\mathcal{N} = 4$ non-commutative super Yang-Mills theory

Masanori HANADA<sup>a,b,c</sup> and Hidehiko SHIMADA<sup>d</sup>

<sup>a</sup>Stanford Institute for Theoretical Physics, Stanford University, Stanford, CA 94305, USA

<sup>b</sup>Yukawa Institute for Theoretical Physics, Kyoto University,  
Kitashirakawa Oiwakecho, Sakyo-ku, Kyoto 606-8502, Japan

<sup>c</sup>The Hakubi Center for Advanced Research, Kyoto University,  
Yoshida Ushinomiya-cho, Sakyo-ku, Kyoto 606-8501, Japan

<sup>d</sup>Okayama Institute for Quantum Physics, Okayama, Japan

## abstract

We study the commutative limit of the non-commutative maximally supersymmetric Yang-Mills theory in four dimensions ( $\mathcal{N} = 4$  SYM). The commutative limits of non-commutative spaces are important in particular in the applications of non-commutative spaces for regularisation of supersymmetric theories (such as the use of non-commutative spaces as alternatives to lattices for supersymmetric gauge theories and interpretations of some matrix models as regularised supermembrane or superstring theories), which in turn can play a prominent role in the study of quantum gravity via the gauge/gravity duality. In general, the commutative limits are known to be singular and non-smooth due to UV/IR mixing effects. We give a direct proof that UV effects do not break the continuity of the commutative limit of the non-commutative  $\mathcal{N} = 4$  SYM to all order in perturbation theory, including non-planar contributions. This is achieved by establishing the uniform convergence (with respect to the non-commutative parameter) of momentum integrals associated with all Feynman diagrams appearing in the theory, using the same tools involved in the proof of finiteness of the commutative  $\mathcal{N} = 4$  SYM.

# 1 Introduction

Non-commutative field theories are non-local deformations of usual local field theories, obtained by replacing products between fields by the so-called Moyal products,

$$f \star g \equiv f e^{\frac{i}{2} \overleftarrow{\partial}_\mu C^{\mu\nu} \overrightarrow{\partial}_\nu} g = fg + \frac{i}{2} C^{\mu\nu} (\partial_\mu f) (\partial_\nu g) + \frac{1}{2!} \left( \frac{i}{2} \right)^2 C^{\mu_1 \nu_1} C^{\mu_2 \nu_2} (\partial_{\mu_1} \partial_{\mu_2} f) (\partial_{\nu_1} \partial_{\nu_2} g) + \dots, \quad (1)$$

where  $C^{\mu\nu} = -C^{\nu\mu}$  are the non-commutativity parameters. Aspects of these theories have been studied from various perspectives in recent years. For a review, see e.g. [1].

In this paper we study the commutative limit,  $C \rightarrow 0$ . The original local field theories are recovered in this limit at the classical level. However, the commutative limit is known to be singular at the quantum level for generic non-commutative field theories, due to an effect called the “UV/IR mixing” [2], as will be elaborated later.

One motivation to study the non-commutative field theory, or the non-commutative space, comes from the expectation that it might provide us with a good mean to regularise, or discretise, quantum theories with infinite degrees of freedom (in particular those with supersymmetry), enabling us to define these theories non-perturbatively. The commutative limit plays a crucial role in this context <sup>1</sup>.

One of the early examples of the application of the non-commutative space is the construction of the matrix model of M-theory as the regularised version of supermembrane theory [3, 4, 5, 6]. (The matrix model has the same amount of supersymmetry as the supermembrane theory; an important advantage of the regularisations using the non-commutative spaces compared to, for example, simple lattice regularisations is that the supersymmetry can often be preserved more easily.) The mathematical structures associated with this regularisation are the same as those appearing in the non-commutative spaces. It is an important issue to understand how one should take the large- $N$  limit of the matrix model at the quantum level, which can be interpreted as the continuum limit of the membrane theory. This continuum limit of the membrane theory is equivalent to the commutative limit  $C \rightarrow 0$ , in the special case where the membrane worldspace (the timeslice of the membrane worldvolume) is given by the so-called non-commutative plane defined by the Moyal product (1). <sup>2</sup> The IKKT matrix model [7] is also obtained by applying a similar regularisation to the superstring worldsheets.

Another example, which is more directly relevant to the subject of this paper, is the application of non-commutative spaces to non-perturbative definitions of supersymmetric Yang-Mills (SYM) theories. The non-perturbative definition of SYM theories via regularisations of them is of course

---

<sup>1</sup>The commutative limit is also important, if one pursues the possibility that our space-time is non-commutative. This non-commutativity is not observed so far. If the commutative limit is continuous, this can be naturally attributed to the smallness of the value of  $C$ . If the commutative limit is singular, which is the case for generic non-commutative field theories, it is more difficult to explain the absence of the non-commutativity in the present observation.

<sup>2</sup>Usually, compact worldspaces of membranes (such as a sphere), whose matrix counterparts are finite dimensional matrices, are considered. The matrix version of a non-compact worldspace by contrast is infinite dimensional because of infinite numbers of degrees of freedom in the IR. Strictly speaking, for the case of infinite dimensional matrices corresponding to the non-compact membranes, the quantum theory is potentially ill-defined due to the infinite number of degrees of freedom, and hence one has to consider it as a certain limit of the theory associated with finite-dimensional matrices. Nonetheless, we believe that at least some of the essential features of the continuum limit of membranes should be captured by the  $C \rightarrow 0$  limit of the non-commutative plane.

a conceptually important theme, and also opens up the possibility of studying non-perturbative properties of these theories via Monte-Carlo simulations. However, construction of satisfactory formulations of regularised SYM theories (in particular those using the lattices) is a notoriously difficult problem, whose general solution is not known to date. In general, one cannot preserve the full supersymmetry algebra in the regularised theory. It is possible to write down a discretised theory which recovers the supersymmetry in the continuum limit at tree level; however, if one goes beyond the tree level, it is in general necessary to introduce counter-terms to prevent the explicit breaking of the supersymmetry via radiative corrections. This procedure is usually called as the fine-tuning. Only for some specific SYM theories, lattice regularisation methods which avoid the fine-tuning problem are known.

A particularly important four-dimensional SYM theory is that with the maximal amount of supersymmetry,  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory (4d  $\mathcal{N} = 4$  SYM hereafter), whose non-perturbative properties are studied extensively in particular in the context of the AdS/CFT correspondence [8]. For the  $\mathcal{N} = 4$  SYM, no formulation based purely on the lattice regularisation is available which is free from the fine-tuning problem<sup>3</sup>. Several years ago, one of the authors proposed a fine-tuning free non-perturbative regularisation of the theory utilising the non-commutative space [10]<sup>4</sup>. In this formalism, two spatial dimensions are non-commutative, being embedded into the gauge degrees of freedom [14, 15]. The remaining two directions are regularised by the lattice method<sup>5</sup>. According to this proposal, after an appropriate continuum limit is taken, one obtains the non-commutative version of the 4d  $\mathcal{N} = 4$  SYM (in which only two spacelike directions are non-commutative). A crucial assumption here is that 4d  $\mathcal{N} = 4$  SYM is obtained as the commutative limit of its non-commutative cousin.

However, it is a non-trivial issue whether one recovers the original theory by taking the commutative limit. An important feature of generic non-commutative field theories is that computations of some Feynman diagrams, whose counterparts in the commutative theory are UV-divergent, yield terms which behave singularly in the  $C \rightarrow 0$  limit. (These terms behave singularly also in the limit where the external momenta go to zero, and hence the appearance of these terms is usually called as the “UV/IR mixing” [2].) This implies that in the commutative limit the observables of a non-commutative field theory are not equivalent to those of its commutative counterpart, at least without further modification of the theory. In this paper we will show that the breaking of the continuity of the commutative limit due to UV effects does not occur for the four-dimensional  $\mathcal{N} = 4$  SYM.<sup>6</sup> More precisely, we prove that the commutative limit is continuous, for all Green functions of the non-commutative  $\mathcal{N} = 4$  SYM in the lightcone gauge, to all order in the perturbation theory, including non-planar contributions. Actually, it was suggested

---

<sup>3</sup>Lattice simulation of  $\mathcal{N} = 4$  SYM, albeit with the parameter fine tuning, is also pursued. See [9] for the latest result.

<sup>4</sup>Prior to this work, a similar technique has been applied for 3d maximal SYM in [11]. See also [12], in which all dimensions are embedded into matrices. Another example which uses non-commutative space can be found in [13].

<sup>5</sup>For two-dimensional super Yang-Mills, there are several proposals of fine-tuning free formulations [16, 10], and there are numerical tests which support the validity of these proposals at the nonperturbative level [17]. For a complete list of references, see a review paper [18].

<sup>6</sup>In this paper, we avoid the introduction of non-commutativity between the timelike and a spacelike coordinate. It has been argued that introduction of the non-commutativity between the time direction and a spacelike direction leads to pathological features, such as the violation of the causality and unitarity [19, 20, 21]. Furthermore, for the regularisation of  $\mathcal{N} = 4$  SYM in the approach of [10], the non-commutativity is introduced only between two spacelike dimensions. It is the commutative limit in this setting that is crucial in this approach.

in [22] that the singular terms are absent for the non-commutative  $\mathcal{N} = 4$  SYM, which implies the continuity of the  $C \rightarrow 0$  limit. One reason behind this suggestion is the well-known finiteness of the commutative ( $C = 0$ )  $\mathcal{N} = 4$  SYM [23, 24]. However, the finiteness of the commutative theory alone does not ensure the continuity, let alone the smoothness, of the commutative limit, as will be explained in detail in section 2. The crucial point is that the finiteness of the original theory is due to non-trivial cancellations between divergent diagrams, and there is no guarantee a priori that these cancellations are not ruined by the introduction of the non-commutativity.

The crucial concept utilised in our proof is the uniform convergence, whose relevance is explained in section 2. Our tools to prove the uniform convergence of all Feynman integrals, the lightcone superspace and power-counting procedures done in two steps, are those used in the original proofs of the finiteness of the commutative theory [23, 24]. We will prove our theorem by showing that these tools remain effective after modifications due to the non-commutativity. This strategy is the same as that taken in the proof of finiteness of the so-called  $\beta$ -deformed  $\mathcal{N} = 4$  SYM [25, 26], and the technical part of our proof is also similar to those given there, though there are a few important differences. This is because the  $\beta$ -deformation is also defined by replacing the ordinary product of the original theory by a  $*$ -product which shares important properties with the Moyal product.

We emphasise that what we show is not merely the finiteness of the non-commutative  $\mathcal{N} = 4$  SYM: the non-commutative SYM is finite, and the finite result is continuous with respect to  $C$ . This is achieved because our power counting procedures ensure the uniform convergence, which is stronger than the mere convergence of Feynman integrals. The finiteness of the non-commutative  $\mathcal{N} = 4$  SYM was proved in [27].

We notice that the application of Weinberg's theorem [28] in the lightcone gauge involves some subtlety as first pointed out in [29]. This point will be discussed later in this paper.

This paper is organised as follows. In section 2, we elaborate on the non-triviality of the commutative limit, and show that the finiteness of the commutative version alone does not imply the smoothness of the commutative limit. The proof of our theorem is given in section 3. We conclude with some discussion in section 4. In an appendix we give an explicit one-loop computation of two point functions in the superfield formulation, which is indeed continuous in the  $C \rightarrow 0$  limit.

## 2 Commutative limit and uniform convergence

In this section, we discuss why the commutative limit is nontrivial, in particular for the  $\mathcal{N} = 4$  SYM. We begin by recalling the so-called UV/IR mixing in generic non-commutative field theories. In terms of the Feynman rules, the effect of the replacement of usual products by the Moyal products (1) simply amounts to introduction of phase factors,

$$e^{-\frac{i}{2}p_\mu C^{\mu\nu} p'_\nu} \quad (2)$$

for each vertex, where  $p$  and  $p'$  are momenta associated with the (external or internal) lines connected to the vertex.

It was then found by explicit one-loop calculations that some Green functions exhibit new types of singularities [2, 30, 31]. These singularities appear only for non-planar diagrams.<sup>7</sup>

---

<sup>7</sup>For planar diagrams, the dependence of the phase factors on the loop momenta cancels out, and hence we

Original UV divergences associated with non-planar diagrams are tamed by the rapid oscillations introduced by the phase factors (2) [32]. One can interpret this as an introduction of an effective UV cut-off, for the internal momenta, of order  $\frac{1}{|Ck|}$ . Here  $k$  is given by some linear combination of external momenta which may be different for different diagrams. The integral is finite and behaves singularly in the limit  $C \rightarrow 0$ . Schematically, they behave like

$$\frac{1}{(C^{\mu\nu}k_\nu)^2} \quad (3)$$

for diagrams which are originally quadratically divergent. For diagrams which are originally logarithmically divergent, the singular behaviour is,

$$\log(C^{\mu\nu}k_\nu)^2. \quad (4)$$

The appearance of these terms is usually called the “UV/IR mixing”: they arise from originally UV divergent graphs, and have singular behaviour in the IR, i.e. when the external momenta are sent to zero.

These terms are also singular in the commutative limit,  $C \rightarrow 0$ , which is the subject of this paper. These singular behaviours arise although the integrands of the Feynman integrals are smooth with respect to  $C$  as in (2). Thus, the  $C \rightarrow 0$  limit and the integral do not commute. More precisely, the  $C \rightarrow 0$  limit and the limit in which the upper bound of the momentum integral is taken to infinity do not commute.

We will prove that these two limits do commute for the case of the  $\mathcal{N} = 4$  model. One might be tempted to think that because the singularities at  $C \rightarrow 0$  originate in the UV divergences, UV-finite theories including 4d  $\mathcal{N} = 4$  SYM admit continuous  $C \rightarrow 0$  limits. In fact this reasoning is not sufficient to ensure the continuity. The point missed in this argument is that the finiteness of the original theory is a result of cancellations of, say, logarithmically divergent diagrams. The introduction of the phase factors may tame different divergent diagrams in different ways (with different effective cut-offs for different diagrams). If this happens, we will have sums of terms of the form  $\log(Ck)^2$ , where  $k$  can be different for various diagrams. Such sums can behave singularly in the limit  $C \rightarrow 0$ . In other words, the original cancellation is ruined. In this manner, a theory which is finite at  $C = 0$  can have a non-smooth  $C \rightarrow 0$  limit. Simple one-dimensional integrals (8)-(12) with this property are presented at the end of this section.

Explicit one-loop computations for super Yang-Mills theories with lower supersymmetry were performed in [33, 34, 35]. In these computations UV/IR mixing terms are found, even for some  $\mathcal{N} = 2$  UV finite theories. Properties of non-commutative  $\mathcal{N} = 2$  theories related to UV/IR mixing are discussed in [36]. For the  $\mathcal{N} = 4$  theory no UV/IR mixing terms are found [33, 37, 38] at the one-loop level.

One can look at this problem of the commutative limit from a more mathematical point of view. It is well-known that what guarantees the validity of the exchange of two limits is the condition of uniform convergence; the cancellation of the divergence at  $C = 0$  does not imply a smooth behaviour for  $C \rightarrow 0$ .

---

have the same Feynman integrals (which may be divergent) as the original commutative theory, multiplied only by over-all phase factors depending on the external momenta. In [27], an argument is given to show the finiteness of the non-commutative  $\mathcal{N} = 4$  SYM focusing only on planar diagrams (since contributions from non-planar diagrams are finite). In order to discuss the singularities in the  $C \rightarrow 0$  limit, one has to study the non-planar diagrams.

Let us recall the definition of uniform convergence. We consider an integral of the form

$$\lim_{\Lambda \rightarrow \infty} \int^{\Lambda} f(p, C) dp = F(C) \quad (5)$$

where  $\Lambda$  is the upper bound of the momentum integral <sup>8</sup>. The definition of uniform convergence with respect to the parameter  $C$  is

$$\text{for } \forall \varepsilon > 0, \quad \exists \Lambda_0 \quad \text{s.t.} \quad \left| \int^{\Lambda} f(p, C) dp - F(C) \right| < \varepsilon \quad \text{for } \forall \Lambda > \Lambda_0 \text{ and } \forall C, \quad (6)$$

while the definition of the usual convergence (for each fixed value of  $C$ ) is given by the condition,

$$\text{for } \forall \varepsilon > 0, \quad \exists \Lambda_0 \quad \text{s.t.} \quad \left| \int^{\Lambda} f(p, C) dp - F(C) \right| < \varepsilon \quad \text{for } \forall \Lambda > \Lambda_0. \quad (7)$$

(Here  $\varepsilon$  is the error in the computation of the total integral  $F(C)$ ; if one wishes to compute  $F(C)$  within this error then one has to choose the upper bound of the integral,  $\Lambda$ , to be larger than the value  $\Lambda_0$ . Thus this  $\Lambda_0$  can be thought of as a quantity which measure the slowness of the convergence; larger  $\Lambda_0$  implies slower convergence.) The only difference between the two definitions is the extra “ $\forall C$ ” in the former: for the usual convergence,  $\Lambda_0$  may depend on  $C$ , whereas for the uniform convergence  $\Lambda_0$  does not depend on  $C$ . In case of the non-uniform convergence,  $\Lambda_0$  can be singularly large for certain values of  $C$ , which makes it possible for  $F(C)$  to develop singularities or discontinuities at these points even for smooth  $f(p, C)$ . If the condition of the uniform convergence is satisfied, smoothness properties of the integrand  $f(p, C)$  with respect to  $C$ , such as the continuity, transfer to that of the  $F(C)$ . In particular, we will use the theorem <sup>9</sup> which states that if  $f(p, C)$  is continuous in both  $p$  and  $C$  and if the convergence is uniform,  $F(C)$  is also continuous in  $C$ .

We conclude this section by illustrating our argument above by some simple one-dimensional integrals. We start from the simplest logarithmically divergent integral

$$\int_{\mu}^{\Lambda} \frac{dp}{p} = \log \frac{\Lambda}{\mu}. \quad (8)$$

Here the parameter  $\mu$  plays the role of the IR cutoff. Introducing the phase factor, we consider an integral

$$\int_{\mu}^{\infty} \frac{1}{p} e^{ipCk} dp \sim -\log(\mu Ck) \quad (9)$$

The right hand side is the leading behaviour for  $\mu Ck \ll 1$ , which can be derived by using a transformation of the integration variable similar to that presented below. For definiteness, we assume  $C > 0, k > 0$ . The logarithmically divergent integral is now tamed by the oscillating phase

---

<sup>8</sup>We note that this  $\Lambda$  has a slightly different character compared to the usual UV cutoff in quantum field theory. Usually one introduces the UV cutoff to make sense out of a divergent integral, to define the (perturbation) theory. It is a non-trivial issue which type of the cutoff procedure (such as the simple cut-off, the Pauli-Villars method) one should employ. For the  $\mathcal{N} = 4$  SYM, since the integrals are finite the type of the cutoff is hardly an issue. We note that the integrals are made finite by suitable combinations of divergent diagrams.

<sup>9</sup>See, for example, [39].

factor to yield a finite result, which behaves singularly in the limit  $Ck \rightarrow 0$ . This is analogous to the “UV/IR mixing” for generic non-commutative field theories.

Now let us present an example which yields a finite result when putting  $C = 0$  in the integrand, but nonetheless have a discontinuity in the  $C \rightarrow 0$  limit,

$$F(C) = \lim_{\Lambda \rightarrow \infty} \int_{\mu}^{\Lambda} \left( \frac{1}{p} e^{ipCk} - \frac{1}{p} e^{ipCk'} \right) dp. \quad (10)$$

If one puts  $C = 0$  in the integrand, or equivalently, if one takes the  $C \rightarrow 0$  limit before the  $\Lambda \rightarrow \infty$  limit, one gets,

$$F(0) = \int_{\mu}^{\infty} \left( \frac{1}{p} - \frac{1}{p} \right) dp = 0, \quad (11)$$

which is of course finite. Meanwhile, one can evaluate the integral (10) by using simple transformations of variables  $u = pCk$  and  $u = pCk'$ ,

$$F(C) = \int_{\mu Ck}^{\mu Ck'} \frac{1}{u} e^{iu} du \sim \log \left( \frac{k'}{k} \right). \quad (12)$$

Again, the right hand side is the leading behaviour when  $C$  is small. Thus  $F(C)$  has a discontinuity at  $C = 0$  and the  $C \rightarrow 0$  limit is not smooth. This corresponds to the dangerous situation, where the cancellation between would-be divergent terms are ruined by the phase factors, yielding singular  $C \rightarrow 0$  limit. We will rule out occurrence of analogous situations for Feynman integrals in the  $\mathcal{N} = 4$  SYM model in the next section.

### 3 The proof

Our proof is technically similar to those given in [25, 26], and in [24], and our main focus will be on differences in particular on the manner uniform convergence is achieved. We will be only concerned with the UV properties of the Feynman integral. We shall assume below that there is an implicit IR cut-off to avoid any possible complication from IR divergences. We note that the phase factor associated with the Moyal product does not introduce the rapid oscillations in the IR (when the loop momenta are small), and hence it seems likely that the structure of IR divergences is not affected much by the non-commutativity.

The outline of the proof is as follows. In section 3.1, we formulate the non-commutative  $\mathcal{N} = 4$  SYM in terms of the lightcone superfield. Due to properties of the Moyal product such as the associativity, the result is quite simple: one replaces products between superfields in the superspace action for the commutative theory with Moyal products.

In section 3.2, we evaluate (an upper bound for) the superficial degree of divergence  $D$  of Feynman integrals. This is done in two steps. In the first step, we make a “rough estimate” of  $D$ , by using techniques of evaluating superfield Feynman graphs similar to those introduced in [40] for  $\mathcal{N} = 1$  supergraphs. At this stage one concludes that  $D \sim 0$ . In the next step, one focusses on vertices connected to external lines; using the particular form of the vertices, one can improve the rough estimate of  $D$  to show that  $D$  is in fact negative.<sup>10</sup>

---

<sup>10</sup>We note that in the lightcone gauge there is no wavefunction renormalisation. In some gauge there is wavefunction renormalisation which does not affect physical observables of the theory.

Finally, we use Weinberg's theorem [28] in section 3.3. In our context Weinberg's theorem implies the uniform convergence, which in turn results in the continuity of the result of the Feynman integrals with respect to  $C$ .

### 3.1 Non-commutative $\mathcal{N} = 4$ SYM in lightcone superspace

In this section we introduce the lightcone superfield formalism in the non-commutative space, which is a natural extension of the original formulation in the commutative space [41].

We define the lightcone coordinates

$$x^\pm = \frac{x^0 \pm x^3}{\sqrt{2}}, \quad (13)$$

where  $x^+$  plays the role of the time coordinate. The remaining two coordinates  $x^1, x^2$  are non-commutative,

$$x^1 \star x^2 - x^2 \star x^1 = iC^{12} = iC. \quad (14)$$

Our metric convention is  $\eta^{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ , and the lightcone components of the metric are  $\eta_{+-} = \eta_{-+} = \eta^{+-} = \eta^{-+} = -1, \eta_{++} = \eta_{--} = \eta^{++} = \eta_{--} = 0$ . We use indices  $\mu, \nu = 0, \dots, 3$  for spacetime coordinates. The lightcone components of the gauge fields are  $A_\pm = (A_0 \pm A_3)/\sqrt{2}$ . We impose the lightcone gauge condition

$$A_- = 0. \quad (15)$$

In this gauge,  $A_\pm$  are not propagating.

There are eight bosonic propagating degrees of freedom: two transverse components of the gauge field  $A^1$  and  $A^2$ , which we combine into a complex field  $A = \frac{1}{\sqrt{2}}(A^1 + iA^2)$  and  $\bar{A} = \frac{1}{\sqrt{2}}(A^1 - iA^2)$  and three complex scalar fields,  $\varphi_{mn} = -\varphi_{nm}$  ( $m, n = 1, \dots, 4$ ) with the condition  $\varphi_{mn} = \epsilon_{mnpq}\varphi^{pq}/2$ , where  $\epsilon_{mnpq}$  is a totally antisymmetric tensor with  $\epsilon_{1234} = +1$ . Half of the spinor fields are not propagating in the lightcone gauge, and there are four complex (single-component) fermions  $\chi^m$  ( $m = 1, \dots, 4$ ).

The action in the lightcone gauge is obtained from the original action by eliminating non-dynamical degrees of freedom such as  $A_+$  [41]. In this procedure the trace cyclicity of the matrix product plays an essential role. The procedure goes through similarly in the non-commutative case, because the Moyal product also satisfies the cyclicity inside the trace<sup>11</sup>,

$$\int (f_1 \star f_2 \star \dots \star f_n) d^d x = \int (f_2 \star \dots \star f_n \star f_1) d^d x. \quad (16)$$

The invariance under supersymmetry is also preserved in a similar way.

Now let us introduce the superfield formulation. There are four bosonic and eight fermionic coordinates,  $x^+, x^-, z = (x^1 + ix^2)/\sqrt{2}, \bar{z} = (x^1 - ix^2)/\sqrt{2}$  and  $\theta^m, \bar{\theta}_m$  ( $m = 1, \dots, 4$ ). Eight kinematical (manifest) supersymmetries are generated by

$$Q^m = -\frac{\partial}{\partial \bar{\theta}_m} - \frac{i}{\sqrt{2}}\theta^m \partial_-, \quad \bar{Q}_m = \frac{\partial}{\partial \theta^m} + \frac{i}{\sqrt{2}}\bar{\theta}_m \partial_-. \quad (17)$$

---

<sup>11</sup>This may be understood as a consequence of the mapping between functions in the non-commutative space and matrices [14]: the integral and the non-commutative product are identified to the trace and the product of matrices, respectively.



The superspace chiral derivatives are defined by

$$d^m = -\frac{\partial}{\partial \bar{\theta}_m} + \frac{i}{\sqrt{2}} \theta^m \partial_-, \quad \bar{d}_m = \frac{\partial}{\partial \theta^m} - \frac{i}{\sqrt{2}} \bar{\theta}_m \partial_-. \quad (18)$$

The scalar superfield  $\Phi$  and its hermitian conjugate  $\bar{\Phi}$  satisfy the chirality condition

$$d^m \Phi = 0, \quad \bar{d}_m \bar{\Phi} = 0 \quad (19)$$

and

$$\bar{\Phi} = \frac{\bar{d}^4}{2\partial_-^2} \Phi, \quad \Phi = \frac{d^4}{2\partial_-^2} \bar{\Phi} \quad (20)$$

We use abbreviations such as,

$$\bar{d}^4 = \bar{d}_1 \bar{d}_2 \bar{d}_3 \bar{d}_4 = \frac{1}{24} \epsilon^{mnpq} \bar{d}_m \bar{d}_n \bar{d}_p \bar{d}_q. \quad (21)$$

This convention differs from those of [41, 24, 25, 26] by a factor of 24. The definition of the superfields remains the same for the non-commutative case since derivatives in the lightcone coordinates commute with  $\star$ -products.

In terms of the component fields, the scalar superfield  $\Phi$  is expressed as

$$\begin{aligned} \Phi(x, \theta, \bar{\theta}) = & -\frac{1}{\partial_-} A(X) - \frac{i}{\partial_-} \theta^m \bar{\chi}_m(X) + \frac{i}{\sqrt{2}} \theta^m \theta^n \bar{\varphi}_{mn}(X) \\ & + \frac{\sqrt{2}}{6} \epsilon_{mnpq} \theta^m \theta^n \theta^p \chi^q(X) - \frac{1}{12} \epsilon_{mnpq} \theta^m \theta^n \theta^p \theta^q \partial_- \bar{A}(X). \end{aligned} \quad (22)$$

Here  $X$  is the chiral coordinate  $X = (x^+, y^-, z, \bar{z})$  where  $y^- \equiv x^- - \frac{i}{\sqrt{2}} \theta^m \bar{\theta}_m$ .

The action in terms of superfields is <sup>12,13</sup>

$$\begin{aligned} S = \frac{1}{8} \int d^4 x \int d^4 \theta d^4 \bar{\theta} \text{Tr} \left\{ -2\bar{\Phi} \frac{\square}{\partial_-^2} \Phi + \frac{8ig}{3} \left( \frac{1}{\partial_-} \bar{\Phi} \cdot [\Phi, \bar{\partial} \Phi]_\star + \frac{1}{\partial_-} \Phi \cdot [\bar{\Phi}, \partial \bar{\Phi}]_\star \right) \right. \\ \left. + 2g^2 \left( \frac{1}{\partial_-} [\Phi, \partial_- \Phi]_\star \cdot \frac{1}{\partial_-} [\bar{\Phi}, \partial_- \bar{\Phi}]_\star + \frac{1}{2} [\Phi, \bar{\Phi}]_\star [\Phi, \bar{\Phi}]_\star \right) \right\} \end{aligned} \quad (23)$$

where the star commutator between two fields  $A, B$  is defined by

$$[A, B]_\star = A \star B - B \star A. \quad (24)$$

The action of the non-commutative SYM is the same as the original theory in the commutative space, except for the replacement of the product with the Moyal product. This is similar to the formulation of  $\beta$ -deformed  $\mathcal{N} = 4$  SYM in terms of the lightcone superfield discussed in [25, 26].

<sup>12</sup>We use the prescription by Mandelstam [23] when defining factors such as  $\frac{1}{\partial_-}$ , which enables us to perform the Wick rotation.

<sup>13</sup>We follow the notation used in [41],  $\partial = (\partial^1 + i\partial^2)/\sqrt{2}$ ,  $\bar{\partial} = (\partial^1 - i\partial^2)/\sqrt{2}$ .

### 3.2 Power counting

In this section, we consider the superficial degree of divergence  $D$ . In usual field theory,  $D$  is determined by counting the powers of momenta. In the non-commutative theory the integrand depends on the momenta non-polynomially due to the phase factors introduced by the  $\star$ -product. We define  $D$  neglecting the phase factors. The superficial degree of divergence so defined is useful when we apply Weinberg's theorem as we will see in section 3.3.

The power counting procedure is divided into two steps. In the first step a “rough” estimate of  $D$  is made, which is refined in the second step. The starting point of the first step is to write down the superspace Feynman rules. The propagator is given by

$$\begin{aligned} & \langle \Phi_{p(1)}^u{}_v(\theta_{(1)}, \bar{\theta}_{(1)}) \Phi_{p(2)}^r{}_s(\theta_{(2)}, \bar{\theta}_{(2)}) \rangle \\ &= \frac{1}{(2\pi)^4} \delta^4(p_{(1)} + p_{(2)}) \times \delta^u{}_s \delta^r{}_v \times \frac{i}{p_\mu^2} d_{(1)}^4 \delta^8(\theta_{(1)} - \theta_{(2)}), \end{aligned} \quad (25)$$

where

$$\delta^8(\theta - \theta') = (\theta - \theta')^4 (\bar{\theta} - \bar{\theta}')^4. \quad (26)$$

Our convention is  $(\theta)^4 = (1/24)\epsilon_{mnpq}\theta^m\theta^n\theta^p\theta^q$ . The vertices can be read off from the action by using the formula (20).

We will now sketch the supergraph power-counting. For details, see [24, 25, 40, 42]. When evaluating a Feynman diagram, one first performs the  $\theta$ -integrals. Focussing on a single internal line, one can get rid of  $d$ 's originating from the propagator and  $\bar{d}$ 's originating from vertices by using partial integration, ending up with a bare superspace  $\delta$ -function. Then the  $\theta$ -integral can be performed, eliminating one  $\theta$  variable. This procedure is to be repeated to the point where only one  $\theta$ -integral is left. In this process, for each loop, one has to use the following identity once:

$$\delta^8(\theta_{(1)} - \theta_{(2)}) d_{(1)}^4 \bar{d}_{(1)}^4 \delta^8(\theta_{(1)} - \theta_{(2)}) = \delta^8(\theta_{(1)} - \theta_{(2)}). \quad (27)$$

Other combinations of two  $\delta$ -functions and chiral derivatives vanish under the  $\theta$ -integral [40, 24]. This means that we lose 4 powers of momentum<sup>14</sup> for each loop. This cancels the original 4 powers of momentum from the loop integral. Thus, the contribution of each loop to the superficial degree of divergence  $D$  is zero.<sup>15</sup> The contribution of the propagator to  $D$  comes from the  $\frac{d^4}{p^2}$  part of (25) and is zero. The contributions of the vertices are also zero as can be read off from the action (23).

The result of the first step in the power counting procedure is thus  $D \sim 0$ . At this stage, we are not distinguishing the external and internal momenta. In the second step, we distinguish them, focussing on a vertex attached to an external line. By certain manipulations using the explicit form of the vertices one can then show that the superficial degree of divergence decreases by one (or more). These manipulations are, (a) moving  $d$ 's or  $\bar{d}$ 's from internal lines to external

<sup>14</sup>We note that  $d$ 's or  $\bar{d}$ 's should be thought of as a square root of momenta, in the power counting procedure.

<sup>15</sup>For the  $\beta$ -deformed theory [25, 26] the equality (27) is modified except for planar diagrams, because the  $\star$ -product for the  $\beta$ -deformation acts on the  $\theta$ -space. Hence the analysis was restricted to the planar level for the  $\beta$ -deformed theory. In the present case, this equality remain unchanged, because the Moyal product does not act on the  $\theta$ 's, so that our analysis is valid for all diagrams including non-planar ones.

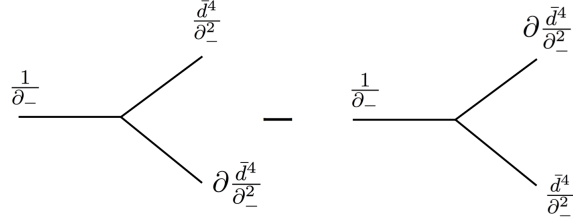


Figure 1: Three vertex (28).

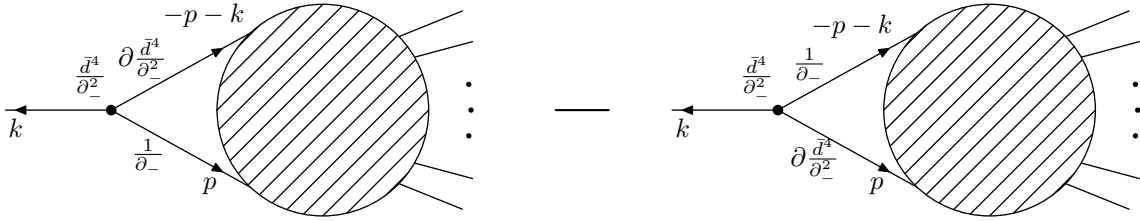


Figure 2: A class of generic processes in which the vertex (28) is involved. The cancellation between two sets of diagrams takes place as shown in (29).

lines via partial integrations (b) cancellations between contributions from different vertices and contractions, in the leading behaviour when the internal momenta are much larger than the external momenta. One has to do this analysis for all possible contractions of all three-point and four-point vertices. Here, one has to verify that the cancellations used in this step occur among contractions which acquires the same phase factors from the non-commutativity.

This step is parallel to the corresponding step in the proof of finiteness of the  $\beta$ -deformed theory given in [25], and we will not give the details in this paper. The present case is actually simpler since the chiral derivatives commute with the Moyal products. In order to illustrate the procedure let us discuss one particular example of the arguments used in this step, for the three-point vertex,

$$\frac{ig}{3} \int d^4x \int d^4\theta d^4\bar{\theta} \text{Tr} \left( \frac{1}{\partial_-} \Phi \cdot [\bar{\Phi}, \partial \bar{\Phi}]_{\star} \right) = \frac{ig}{12} \int d^4x \int d^4\theta d^4\bar{\theta} \text{Tr} \left( \frac{1}{\partial_-} \Phi \cdot \left[ \frac{\bar{d}^4}{\partial_-^2} \Phi, \partial \frac{\bar{d}^4}{\partial_-^2} \Phi \right]_{\star} \right), \quad (28)$$

which can be represented diagrammatically as in Fig. 1. In our convention products of fields are always taken to be counter-clockwise in Feynman diagrams. The contributions we consider are shown in Fig. 2. The shaded disk represents general processes. We are focussing on a particular vertex connected to the external line, which is given by Fig. 1. The contractions we consider are shown in Fig. 2. By moving  $\bar{d}^4$  appropriately, it is possible to show that the contributions from two diagrams are the same except for the momentum factor. The sum of these momentum factors are

$$-\frac{p+k}{k_-^2 p_- (p+k)_-^2} + \frac{p}{k_-^2 p_-^2 (p_- + k_-)}. \quad (29)$$

The leading terms for  $p \gg k$  cancel out. This cancellation implies that the superficial degree of

divergence is decreased by one, improving the convergence. It is essential that the cancellation is not affected by the phase factors originating in the Moyal product; the phase factors associated with the two vertices shown in Fig. 2 are identical.

As discussed in [25], there are a few exceptional diagrams in which general arguments do not apply. They are one-loop diagrams and are evaluated explicitly in the appendix.

### 3.3 Weinberg's theorem, UV finiteness and the commutative limit

At each order in perturbation theory, we have finite sums of terms of the form

$$\int f(p, C) dp = \int g(p) e^{i \sum_{(i)} p_{(i)\mu} C^{\mu\nu} p'_{(i)\nu}} dp. \quad (30)$$

The integrand is given by a rational function  $g(p)$  multiplied by a single phase factor. The arguments in the previous section shows that the superficial degree divergence of  $\int g(p) dp$  is negative. (We recall that our definition of  $D$  does not include the phase factor.) Here  $p_{(i)}$ , and  $p'_{(i)}$  are some linear combinations of the internal and external momenta.

It is also easy to see that the same holds for all sub-diagrams. One can now invoke Weinberg's theorem<sup>16</sup>, which assures the absolute convergence of the integral  $\int g(p) dp$ , i.e. the convergence of the integral

$$\int |g(p)| dp. \quad (31)$$

A quick way to see that (31) implies the uniform convergence of (30) is the following. The convergence of (31), or the absolute convergence of (30), guarantees that the original integral (30) converges to a definite value,  $F(C)$ . Then,  $\left| \int^\Lambda f(p, C) dp - F(C) \right|$  in the condition for uniform convergence, (6), can be rewritten as<sup>17</sup>

$$\left| \int_\Lambda^\infty f(p, C) dp \right| = \left| \int_\Lambda^\infty g(p) e^{i \sum p C p'} dp \right|, \quad (32)$$

and the r.h.s. satisfies the elementary inequality

$$\left| \int_\Lambda^\infty g(p) e^{i \sum p C p'} dp \right| < \int_\Lambda^\infty |g(p)| dp. \quad (33)$$

Because of the convergence of (31), for arbitrary  $\varepsilon > 0$  there exists  $\Lambda_0$  such that for any  $\Lambda > \Lambda_0$   $\int_\Lambda^\infty |g(p)| dp < \varepsilon$ . By using the same  $\varepsilon$  and  $\Lambda$  we have

$$\left| \int^\Lambda f(p, C) dp - F(C) \right| = \left| \int_\Lambda^\infty f(p, C) dp \right| < \varepsilon \quad (34)$$

<sup>16</sup>The assumption of Weinberg's theorem is that the superficial degree of divergence is negative for all possible linear subspaces in the integration variables [28]. For Lorentz invariant Feynman integrals (after Wick rotation), the denominator depends on the momenta always in the form  $p_\mu^2$ , and therefore it is sufficient to consider all possible subgraphs to guarantee that this requirement is met. In the lightcone gauge, there are factors of  $1/\partial_-$  in the integrand. As a consequence, it is necessary to separately examine the linear subspaces distinguishing longitudinal and transverse components for all loop momenta. For example, one should consider the region in which transverse components are sent to infinity but longitudinal components are kept finite. This was not done in the original finiteness proof of 4d  $\mathcal{N} = 4$  theory in [24]. (For the proof given in [23], this subtlety is pointed out and a resolution of it is discussed in [29].) We will make a few comments on this subtlety also in section 4.

<sup>17</sup>We use the notation  $\int_\Lambda^\infty$  to denote the (multi-dimensional) integral complementary to  $\int^\Lambda$ , i.e.  $\int^\infty = \int^\Lambda + \int_\Lambda^\infty$ .

for arbitrary  $C$ , which is the condition of uniform convergence.

We now use the theorem which states that if  $f(p, C)$  is continuous in  $p$  and  $C$  and  $\Lambda \rightarrow \infty$  is uniformly convergent,  $F(C)$  is continuous in  $C$  [39]. Thus we have shown that there is no discontinuity in  $C$ , in particular for  $C \rightarrow 0$ .

## 4 Conclusion and discussion

In this paper we have analysed the UV properties of the non-commutative version of the 4d  $\mathcal{N} = 4$  SYM. We have shown that the cancellations between diagrams at  $C = 0$  i.e. of the commutative theory, which are responsible for the finiteness of the commutative theory, persist for the non-commutative theory as well. These cancellations ensure that the momentum integrals converge uniformly with respect to  $C$ , which in turn implies that the Green functions (in the lightcone gauge) have no discontinuity in  $C$ , to all order in perturbation theory.

This continuity is one of the key steps of the non-perturbative definition of 4d  $\mathcal{N} = 4$  SYM proposed in [10]. (The proposal for 3d maximal SYM [11] also includes implicitly the assumption that the commutative limit is continuous. In this case, however, our proof does not directly apply because we utilised the independence of lightcone coordinates and the Moyal product; in three dimension, in the presence of the non-commutativity in two space-like directions, the lightcone coordinates become non-commutative inevitably.) The proposal may eventually enable us to study nonperturbative features of 4d  $\mathcal{N} = 4$  SYM numerically, which should deepen our understanding of the AdS/CFT correspondence and may make it possible to study quantum aspects of gravity from a dual gauge theory. Such numerical approach has been so far successful for the  $(0+1)$ -d theory [6, 5][43] (for recent work see [44]) and  $(1+1)$ -d theory [45]. The 4d theory, which has been considered much more extensively in the past, would serve as an even better laboratory.

The non-commutative 4d  $\mathcal{N} = 4$  SYM is also an interesting theory in its own right. It is believed that this theory has a gravity dual, and aspects of the duality have been studied for example in [46, 47, 48].

The appearance of singularities of the form (3) and (4), is a characteristic feature of non-commutative theories.<sup>18</sup> These singularities play important roles in the study of the relations between non-commutative field theories and ordinary field theories.<sup>19</sup> They may also have some relevance in some proposals discussing relation between non-commutative field theories and gravity [52, 53]. We stress that our work is the first to establish strong constraints on these singularities to the all order in perturbation theory. It would be interesting to consider more general theories and properties from this approach.

We believe that our analysis of the continuity of Green functions with respect to the non-commutative parameter  $C$  will be the core in the study of the commutative limit in the  $\mathcal{N} = 4$

---

<sup>18</sup>Usually, the singular terms (3) and (4) are characterised by the singular behaviour in the IR region of the external momenta. What we have studied in this paper is the smoothness properties when the non-commutative parameter goes to zero. Our analysis alone does not exclude the occurrence of IR singularities, since it is at least logically possible to have a term which behaves singularly when external momenta go to zero but does not have a discontinuity in  $C$ .

<sup>19</sup>For a recent interesting discussion on the relation between the analog of the UV/IR mixing effect on a non-commutative sphere and appearance of a certain non-local interaction in the renormalisation group flow, see [49]. See also [50, 51].

SYM. There are directions in which one can extend our analysis in this paper. First, one can also study the differentiability of Green functions with respect to  $C$  by studying the convergence property of the integral which have the integrand given by the  $C$ -derivative of the original integrand. Each  $C$ -derivative acting on the phase factor  $e^{ipCp'}$  brings in extra two powers of momenta, increasing the superficial degree of convergence. It should be possible to clarify the structure of singularities in the  $C$ -derivatives of Green functions by appropriate extension of our method. Second, we have confined ourselves to study of Green functions of the fundamental fields. It would be also interesting to study gauge invariant operators. Recently correlation function of composite operators have been studied in the lightcone gauge formalism [54]. Third, in this paper we have used the lightcone gauge. Hence, the Lorentz invariance of theory is not manifest. It is natural to expect that the lightcone gauge formulation is equivalent to a covariant formulation, say, in the Lorentz gauge, which leads immediately also to the Lorentz invariance. Although the gauge independence of perturbative gauge theories is fairly well-established, this issue should be more non-trivial for non-commutative gauge theories because of the non-locality introduced by Moyal products. This issue is not unrelated to the issue of the gauge invariant operators. When discussing the gauge independence, one has to fix one's attention on a set of gauge invariant observables. In the standard non-conformal gauge theories, one usually studies the S-matrix. Since the (commutative)  $\mathcal{N} = 4$  theory is conformal, a natural candidate for the set of observables are  $n$ -point correlation functions of composite operators (with definite conformal dimensions).

In this paper, the convergence of the Feynman integrals was studied applying Weinberg's theorem, following previous work [24]. In the lightcone superfield formalism Feynman rules are not manifestly Lorentz invariance in particular because of the appearance of factors of  $\frac{1}{\partial_-}$ . Hence one should examine the limit which breaks the symmetry, for example, a regime in which transverse components are taken to be large while longitudinal components of momenta are kept finite. In appendix A we will show explicitly that there is a more non-trivial cancellation at the one-loop level. It is possible to classify the UV regions for general Feynman diagrams and study the superficial degree of divergence by using appropriate diagrammatic techniques. This issue will be addressed in a separate publication.

We hope that this work can provide a basis for studies of the application of non-commutative  $\mathcal{N} = 4$  SYM and serve as a starting point to clarify the relation between non-commutative field theories and their commutative counterparts more generally.

## Acknowledgements

The authors would like to thank S. Kovacs, S. Matsuura, F. Sugino for their careful reading of our manuscript and very useful discussions and comments. We would like to thank S. Ananth, L. Brink, L. Dixon, T. McLoughlin, and J. Nishimura for very useful discussions and comments. The work of M. H. is supported in part by the Grant-in-Aid of the Japanese Ministry of Education, Sciences and Technology, Sports and Culture (MEXT) for Scientific Research (No. 25287046), and the National Science Foundation under Grant No. PHYS-1066293 and the hospitality of the Aspen Center for Physics.

## A Explicit one-loop computations

In this appendix we perform explicit one-loop computations of two-point Green functions of the non-commutative  $\mathcal{N} = 4$  SYM in the lightcone superfield formalism. The asymmetric asymptotic region – in which the transverse components goes to infinity while longitudinal components remain finite – is important. There are two superfield diagrams. Superficial degree of divergence for each diagram is negative in the usual power counting. However, the diagrams are both logarithmically divergent because of the asymmetric asymptotic region. The divergent contributions from the two diagrams cancel each other and the result behaves well in the  $C \rightarrow 0$  limit.

Below in section A.1 we explain the Feynman rules. The next section A.2 summarises the result for each diagrams and explains the cancellations. In section A.3 we compile some useful formulae.

### A.1 Feynman rules for $\mathcal{N} = 4$ SYM

Because  $\Phi$  and  $\bar{\Phi}$  are related by (20), we can rewrite the action (23) in terms only of  $\Phi$ . We perform a Fourier transformation on the  $x$  coordinates, leaving the  $\theta$  coordinates,

$$\Phi(x) = \int \Phi_p e^{ip \cdot x} d^4 p. \quad (35)$$

Then the superspace propagator can be derived from the quadratic part of the action,

$$\begin{aligned} & \langle \Phi_{p(1)}^u{}_v(\theta_{(1)}, \bar{\theta}_{(1)}) \Phi_{p(2)}^r{}_s(\theta_{(2)}, \bar{\theta}_{(2)}) \rangle \\ &= \frac{1}{(2\pi)^4} \delta^4(p_{(1)} + p_{(2)}) \times \delta^u{}_s \delta^r{}_v \times \frac{i}{p_\mu^2} d_{(1)}^4 \delta^8(\theta_{(1)} - \theta_{(2)}). \end{aligned} \quad (36)$$

Here  $U(N)$  colour indices  $u, v, \dots$  are shown explicitly.

The cubic and quartic vertices are read off from the interaction part of the action,

$$\begin{aligned} iS_{\text{int}} &= \int d^4 x d^8 \theta \text{Tr} \left( -\frac{g}{6} \frac{\bar{d}^4}{\partial_-^3} \Phi \cdot \left[ \Phi, \bar{\partial} \Phi \right]_* - \frac{g}{12} \frac{1}{\partial_-} \Phi \cdot \left[ \frac{\bar{d}^4}{\partial_-^2} \Phi, \partial \frac{\bar{d}^4}{\partial_-^2} \Phi \right]_* \right. \\ &\quad \left. + \frac{ig^2}{16} \frac{1}{\partial_-} \left[ \Phi, \partial_- \Phi \right]_* \cdot \frac{1}{\partial_-} \left[ \frac{\bar{d}^4}{\partial_-^2} \Phi, \frac{\bar{d}^4}{\partial_-} \Phi \right]_* + \frac{ig^2}{32} \left[ \Phi, \frac{\bar{d}^4}{\partial_-^2} \Phi \right]_* \left[ \Phi, \frac{\bar{d}^4}{\partial_-^2} \Phi \right]_* \right). \end{aligned} \quad (37)$$

In momentum space this becomes

$$\begin{aligned} iS_{\text{int}} &= \int d^8 \theta d^4 k d^4 p d^4 q (2\pi)^4 \delta^4(k + p + q) e^{-\frac{i}{2}(k_\mu C^{\mu\nu} l_\nu + p_\mu C^{\mu\nu} q_\nu)} \\ &\quad \times \text{Tr} \left( \frac{g}{6} \frac{\bar{q} - \bar{p}}{k_-^3} (\bar{d}^4 \Phi)_k \Phi_p \Phi_q + \frac{g}{12} \frac{p - q}{k_- p_-^2 q_-^2} \Phi_k (\bar{d}^4 \Phi)_p (\bar{d}^4 \Phi)_q \right) \\ &+ \int d^8 \theta d^4 k d^4 l d^4 p d^4 q (2\pi)^4 \delta^4(k + l + p + q) e^{-\frac{i}{2}(k_\mu C^{\mu\nu} l_\nu + p_\mu C^{\mu\nu} q_\nu)} \\ &\quad \times \text{Tr} \left( \frac{ig^2}{8} \frac{k_- q_- + l_- p_-}{p_-^2 q_-^2 (p + q)_-^2} \Phi_k \Phi_l (\bar{d}^4 \Phi)_p (\bar{d}^4 \Phi)_q + \frac{ig^2}{16} \frac{1}{l_-^2 q_-^2} \Phi_k (\bar{d}^4 \Phi)_l \Phi_p (\bar{d}^4 \Phi)_q \right). \end{aligned} \quad (38)$$

We make it as a rule to write all vertices in a way in which matrix products go in a counter clock-wise order. In this convention, the diagrams can be written unambiguously without the double-line notation, and we do not show the colour indices explicitly in the Feynman rules. One can easily recast the single-line diagrams in our convention as diagrams written in the double-line convention, which in turn is convenient to study the colour structure of the result.

As explained in the main text we perform the  $\theta$ -integral by moving  $d$ 's and  $\bar{d}$ 's via partial integrations, using the identity (27) and

$$\{d^m, \bar{d}_n\} = \sqrt{2}i\delta^m_n \partial_- . \quad (39)$$

The Feynman rules are,

$$\xrightarrow{d^4} p = \frac{i}{p_\mu^2} \quad (40)$$

$$\begin{array}{c} p \\ \swarrow \\ \bullet \\ \searrow \\ q \end{array} \xrightarrow{\bar{d}^4} k = \frac{g}{6} \frac{\bar{q} - \bar{p}}{k_-^3} e^{-\frac{i}{2} p_\mu C^{\mu\nu} q_\nu} \quad (41)$$

$$\begin{array}{c} p \\ \swarrow \\ \bullet \\ \searrow \\ q \end{array} \xrightarrow{\bar{d}^4} k = \frac{g}{12} \frac{p - q}{k_- p_-^2 q_-^2} e^{-\frac{i}{2} p_\mu C^{\mu\nu} q_\nu} \quad (42)$$

$$\begin{array}{c} p \quad l \\ \swarrow \quad \searrow \\ \bullet \\ \swarrow \quad \searrow \\ q \quad k \end{array} \xrightarrow{\bar{d}^4} = \frac{ig^2}{8} \frac{k_- q_- + l_- p_-}{p_-^2 q_-^2 (p + q)_-^2} e^{-\frac{i}{2} (k_\mu C^{\mu\nu} l_\nu + p_\mu C^{\mu\nu} q_\nu)} \quad (43)$$

$$\begin{array}{c} p \quad l \\ \swarrow \quad \searrow \\ \bullet \\ \swarrow \quad \searrow \\ q \quad k \end{array} \xrightarrow{\bar{d}^4} = \frac{ig^2}{16} \frac{1}{l_-^2 q_-^2} e^{-\frac{i}{2} (k_\mu C^{\mu\nu} l_\nu + p_\mu C^{\mu\nu} q_\nu)} \quad (44)$$

Here we used the convention

$$p = \frac{1}{\sqrt{2}}(p^1 + ip^2), \quad \bar{p} = \frac{1}{\sqrt{2}}(p^1 - ip^2). \quad (45)$$



We use indices  $\mu, \nu = 0, 1, \dots, 3$  for four-vectors, and indices  $i, j = 1, 2$  for the transverse components.

## A.2 One loop computation

We shall compute the two-point part of the 1PI effective action. Firstly we define the partition function with a source  $J$  by

$$Z[J] = \int e^{iS + J \cdot \phi} \mathcal{D}\phi. \quad (46)$$

Then by using the vacuum expectation value of  $\phi$ , we introduce the effective action  $\Gamma$  as

$$i\Gamma = \log Z - J \cdot \langle \phi \rangle. \quad (47)$$

The effective action takes as its argument the vacuum expectation value of the quantum field. For simplicity, we shall use in the following the same symbol  $\Phi$  for the vacuum expectation value of the superfield as the argument of the effective action.

One loop contributions to the two-point part of  $i\Gamma$  come from the four diagrams in Fig. 3.

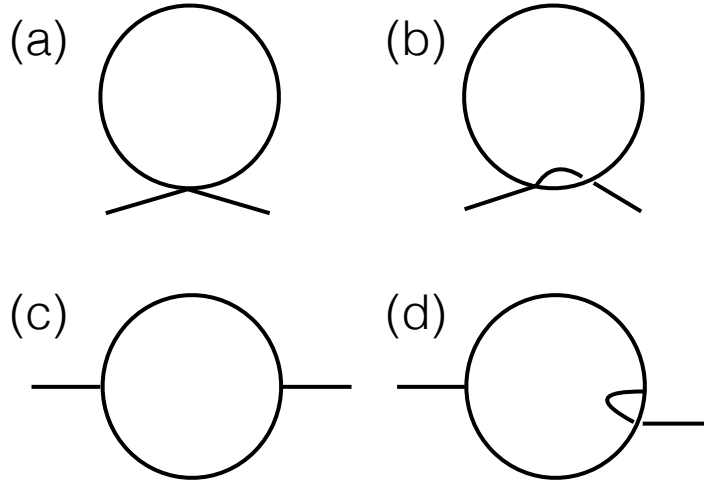


Figure 3: One loop contributions to  $i\Gamma$ .

The contribution to  $i\Gamma$  from the diagram (a) is

$$\int d^4k d^8\theta (2\pi)^4 \text{Tr}(\bar{d}^4\Phi)_k \Phi_{-k} \times \int \frac{d^4p}{(2\pi)^4} g^2 N \frac{1}{p_\mu^2 (p_- + k_-)^2 (p_- - k_-)^2} \quad (48)$$

which is derived by the superfield technique first introduced for  $\mathcal{N} = 1$  superfields in [40], see also [42]. For the case of the  $\mathcal{N} = 4$  theory in the lightcone gauge, see [24, 25]. The superficial degree of divergence is  $-2$  in usual Lorentz invariant power counting. If only transverse momenta

are large, it is 0. If only longitudinal momenta are large, it is  $-4$ . We represent this by saying the superficial degree of divergence is  $(-2, 0, -4)$ . The contribution of this diagram is actually logarithmically divergent, because of the region where transverse components are large and longitudinal components are finite. Actually there are similar logarithmically divergent terms from (c), which cancel the divergent contribution. In order to compute this integral, we rewrite the contribution as

$$\int d^4k d^8\theta (2\pi)^4 \text{Tr}(\bar{d}^4\Phi)_k \Phi_{-k} \times \int \frac{d^4p}{(2\pi)^4} \left( \frac{g^2 N}{4} \right) \times \left( \frac{1}{k_-^2} \frac{1}{p_\mu^2(p_- + k_-)^2} + \frac{1}{k_-^2} \frac{1}{p_\mu^2(p_- - k_-)^2} + \frac{1}{k_-^3} \frac{1}{p_\mu^2(p_- + k_-)} + \frac{-1}{k_-^3} \frac{1}{p_\mu^2(p_- - k_-)} \right). \quad (49)$$

by using a partial fraction expansion.

The contribution from the diagram (b) is,

$$\begin{aligned} & \int d^4k d^8\theta (2\pi)^4 \text{Tr}(\bar{d}^4\Phi)_k \text{Tr}\Phi_{-k} \times \int \frac{d^4p}{(2\pi)^4} (-g^2) e^{ip_\mu C^{\mu\nu} k_\nu} \frac{1}{p_\mu^2(p_- + k_-)^2(p_- - k_-)^2} \\ = & \int d^4k d^8\theta (2\pi)^4 \text{Tr}(\bar{d}^4\Phi)_k \text{Tr}\Phi_{-k} \times \int \frac{d^4p}{(2\pi)^4} \left( \frac{-g^2}{4} \right) e^{ip_\mu C^{\mu\nu} k_\nu} \times \\ & \left( \frac{1}{k_-^2} \frac{1}{p_\mu^2(p_- + k_-)^2} + \frac{1}{k_-^2} \frac{1}{p_\mu^2(p_- - k_-)^2} + \frac{1}{k_-^3} \frac{1}{p_\mu^2(p_- + k_-)} + \frac{-1}{k_-^3} \frac{1}{p_\mu^2(p_- - k_-)} \right) \end{aligned} \quad (50)$$

We note that the contribution from the planar diagram (a) and the non-planar diagram (b) differs simply by a factor  $-\frac{1}{N}$  and the phase factor  $e^{ip_\mu C^{\mu\nu} k_\nu}$  at the level of the integrand, as it should be. The superficial degree of divergence is the same as the corresponding planar diagram (a), i.e.  $(-2, 0, -4)$ .

The contribution from the diagram (c) is

$$\begin{aligned} & \int d^4k d^8\theta (2\pi)^4 \text{Tr}(\bar{d}^4\Phi)_k \Phi_{-k} \times \int \frac{d^4p}{(2\pi)^4} \left( -\frac{g^2 N}{18} \right) \frac{1}{p_\mu^2(p+k)_\mu^2 k_-^2 p_-^2(p_- + k_-)^2} \\ & \quad \times |(2p+k)k_- + (p-k)(p+k)_- - (p+2k)p_-|^2 \\ = & \int d^4k d^8\theta (2\pi)^4 \text{Tr}(\bar{d}^4\Phi)_k \Phi_{-k} \times \int \frac{d^4p}{(2\pi)^4} \left( -\frac{g^2 N}{4} \right) \frac{p_i^2 k_-^2 + k_i^2 p_-^2 - 2k_i p_i k_- p_-}{p_\mu^2(p+k)_\mu^2 k_-^2 p_-^2(p_- + k_-)^2} \end{aligned} \quad (51)$$

The superficial degree of divergence is again  $(-2, 0, -4)$ . This can be rewritten as,

$$\begin{aligned} & \int d^4k d^8\theta (2\pi)^4 \text{Tr}(\bar{d}^4\Phi)_k \Phi_{-k} \times \int \frac{d^4p}{(2\pi)^4} \left( -\frac{g^2 N}{4} \right) \times \\ & \left( \frac{1}{k_-^2} \frac{1}{(p+k)_\mu^2 p_-^2} + \frac{1}{k_-^2} \frac{1}{p_\mu^2(p_- + k_-)^2} \right. \\ & + \frac{-1}{k_-^3} \frac{1}{(p+k)_\mu^2 p_-} + \frac{1}{k_-^3} \frac{1}{p_\mu^2(p_- + k_-)} + \frac{1}{k_-^3} \frac{1}{(p+k)_\mu^2(p_- + k_-)} + \frac{-1}{k_-^3} \frac{1}{p_\mu^2 p_-} \\ & \left. + \frac{k_\mu^2}{k_-^3} \frac{1}{p_\mu^2(p+k)_\mu^2 p_-} + \frac{-k_\mu^2}{k_-^3} \frac{1}{p_\mu^2(p+k)_\mu^2(p_- + k_-)} \right). \end{aligned} \quad (52)$$

Here the key steps in the manipulation are

$$\begin{aligned}
& \frac{p_i^2 k_-^2 + k_i^2 p_-^2 - 2k_i p_i k_- p_-}{p_\mu^2 (p+k)_\mu^2 k_-^2 p_-^2 (p_- + k_-)^2} = \frac{k_- (p_- + k_-) p_i^2 - k_- p_- (p+k)_i^2 + p_- (p_- + k_-) k_i^2}{p_\mu^2 (p+k)_\mu^2 k_-^2 p_-^2 (p_- + k_-)^2} \\
&= \frac{1}{k_-} \frac{p_i^2}{p_\mu^2 (p+k)_\mu^2 p_-^2 (p_- + k_-)} + \frac{1}{k_-} \frac{-(p+k)_i^2}{p_\mu^2 (p+k)_\mu^2 p_- (p_- + k_-)^2} + \frac{1}{k_-^2} \frac{k_i^2}{p_\mu^2 (p+k)_\mu^2 p_- (p_- + k_-)} \\
&= \frac{1}{k_-} \frac{p_\mu^2 + 2p_+ p_-}{p_\mu^2 (p+k)_\mu^2 p_-^2 (p_- + k_-)} + \frac{1}{k_-} \frac{-(p+k)_\mu^2 - 2(p+k)_+ (p+k)_-}{p_\mu^2 (p+k)_\mu^2 p_- (p_- + k_-)^2} + \frac{1}{k_-^2} \frac{k_i^2}{p_\mu^2 (p+k)_\mu^2 p_- (p_- + k_-)} \\
&= \frac{1}{k_-} \frac{1}{(p+k)_\mu^2 p_-^2 (p_- + k_-)} + \frac{-1}{k_-} \frac{1}{p_\mu^2 p_- (p_- + k_-)^2} \\
&\quad + \frac{1}{k_-} \frac{2p_+}{p_\mu^2 (p+k)_\mu^2 p_- (p_- + k_-)} + \frac{-1}{k_-} \frac{2(p+k)_+}{p_\mu^2 (p+k)_\mu^2 p_- (p_- + k_-)} + \frac{1}{k_-^2} \frac{k_i^2}{p_\mu^2 (p+k)_\mu^2 p_- (p_- + k_-)} \\
&= \frac{1}{k_-} \frac{1}{(p+k)_\mu^2 p_-^2 (p_- + k_-)} + \frac{-1}{k_-} \frac{1}{p_\mu^2 p_- (p_- + k_-)^2} + \frac{k_\mu^2}{k_-^2} \frac{1}{p_\mu^2 (p+k)_\mu^2 p_- (p_- + k_-)}. \tag{53}
\end{aligned}$$

The last equality involves cancellations between terms proportional to  $p_+$ .

The contribution from the diagram (d) is

$$\int d^4 k d^8 \theta (2\pi)^4 \text{Tr}(\bar{d}^4 \Phi)_k \text{Tr} \Phi_{-k} \times \int \frac{d^4 p}{(2\pi)^4} \left( \frac{g^2}{4} \right) e^{ip_\mu C^{\mu\nu} k_\nu} \frac{p_i^2 k_-^2 + k_i^2 p_-^2 - 2k_i p_i k_- p_-}{p_\mu^2 (p+k)_\mu^2 k_-^2 p_-^2 (p_- + k_-)^2}, \tag{54}$$

whose superficial degree of divergence is  $(-2, 0, -4)$ . Again, the contribution of (c) and (d) is related simply by a factor of  $-1/N$  and the non-commutative phase at the level of the integrand. By a computation similar to the case of (c), this reduces to

$$\begin{aligned}
& \int d^4 k d^8 \theta (2\pi)^4 \text{Tr}(\bar{d}^4 \Phi)_k \text{Tr} \Phi_{-k} \times \int \frac{d^4 p}{(2\pi)^4} \left( \frac{g^2}{4} \right) e^{ip_\mu C^{\mu\nu} k_\nu} \times \\
& \left( \frac{1}{k_-^2} \frac{1}{(p+k)_\mu^2 p_-^2} + \frac{1}{k_-^2} \frac{1}{p_\mu^2 (p_- + k_-)^2} \right. \\
& + \frac{-1}{k_-^3} \frac{1}{(p+k)_\mu^2 p_-} + \frac{1}{k_-^3} \frac{1}{p_\mu^2 (p_- + k_-)} + \frac{1}{k_-^3} \frac{1}{(p+k)_\mu^2 (p_- + k_-)} + \frac{-1}{k_-^3} \frac{1}{p_\mu^2 p_-} \\
& \left. + \frac{k_\mu^2}{k_-^3} \frac{1}{p_\mu^2 (p+k)_\mu^2 p_-} + \frac{-k_\mu^2}{k_-^3} \frac{1}{p_\mu^2 (p+k)_\mu^2 (p_- + k_-)} \right). \tag{55}
\end{aligned}$$

Combining the planar contributions from (a) and (c), we obtain

$$\begin{aligned}
& \int d^4 k d^8 \theta (2\pi)^4 \text{Tr}(\bar{d}^4 \Phi)_k \Phi_{-k} \times \int \frac{d^4 p}{(2\pi)^4} \left( -\frac{g^2 N}{4} \right) \times \\
& \left( \frac{-1}{k_-^2} \frac{1}{p_\mu^2 (p_- - k_-)^2} + \frac{1}{k_-^2} \frac{1}{(p+k)_\mu^2 p_-^2} \right. \\
& + \frac{1}{k_-^3} \frac{1}{p_\mu^2 (p_- - k_-)} + \frac{-1}{k_-^3} \frac{1}{(p+k)_\mu^2 p_-} + \frac{1}{k_-^3} \frac{1}{(p+k)_\mu^2 (p_- + k_-)} + \frac{-1}{k_-^3} \frac{1}{p_\mu^2 p_-} \\
& \left. + \frac{k_\mu^2}{k_-^3} \frac{1}{p_\mu^2 (p+k)_\mu^2 p_-} + \frac{-k_\mu^2}{k_-^3} \frac{1}{p_\mu^2 (p+k)_\mu^2 (p_- + k_-)} \right). \tag{56}
\end{aligned}$$

By shifting integration variables appropriately,<sup>20</sup> one can show that only the last two terms in the parentheses remain. Hence we have,

$$\int d^4k d^8\theta (2\pi)^4 \text{Tr}(\bar{d}^4\Phi)_k \Phi_{-k} \times \int \frac{d^4p}{(2\pi)^4} \left( -\frac{g^2 N}{4} \right) \times \left( \frac{k_\mu^2}{k_-^3} \frac{1}{p_\mu^2(p+k)_\mu^2 p_-} + \frac{-k_\mu^2}{k_-^3} \frac{1}{p_\mu^2(p+k)_\mu^2(p_-+k_-)} \right). \quad (57)$$

The superficial degree of divergence is  $(-1, -2, -3)$ . Thus we see that the integral is finite due to cancellations between diagrams (a) and (c). The integral can be computed by the method explained in the next subsection. Using (75), this can be rewritten as,

$$\int d^4k d^8\theta (2\pi)^4 \text{Tr}(\bar{d}^4\Phi)_k \Phi_{-k} \times \left( -\frac{g^2 N}{2} \right) \times i\pi^2 \frac{1}{k_-} \int_0^1 dt \frac{1}{t} \log \frac{t(1-t)k_\mu^2}{(tk_\mu^2 - t^2 k_i^2)}. \quad (58)$$

For the combined non-planar contribution to  $i\Gamma$  from (b) and (d), we can perform the same shift without changing the phase factor  $e^{ip_\mu C^{\mu\nu} k_\nu}$ . Then we obtain

$$\int d^4k d^8\theta (2\pi)^4 \text{Tr}(\bar{d}^4\Phi)_k \text{Tr}\Phi_{-k} \times \int \frac{d^4p}{(2\pi)^4} \left( \frac{g^2}{4} \right) e^{ip_\mu C^{\mu\nu} k_\nu} \times \left( \frac{k_\mu^2}{k_-^3} \frac{1}{p_\mu^2(p+k)_\mu^2 p_-} + \frac{-k_\mu^2}{k_-^3} \frac{1}{p_\mu^2(p+k)_\mu^2(p_-+k_-)} \right). \quad (59)$$

The superficial degree of divergence is  $(-1, -2, -3)$ , and the integral is finite by cancellation between diagrams (b) and (d). Therefore, the commutative limit should be continuous. In order to see it explicitly, we rewrite this expression by using the Bessel function of the second kind,

$$(59) = \int d^4k d^8\theta (2\pi)^4 \text{Tr}(\bar{d}^4\Phi)_k \text{Tr}\Phi_{-k} \times \left( \frac{g^2}{2} \right) \times \left( -2i\pi^2 \frac{1}{k_-} \int_0^1 dt \frac{1}{t} \left( K_0 \left( \sqrt{t(1-t)k_\mu^2 \tilde{k}_i^2} \right) - K_0 \left( \sqrt{(tk_\mu^2 - t^2 k_i^2) \tilde{k}_i^2} \right) \right) \right), \quad (60)$$

by using (74). Here  $\tilde{k}^\mu = C^{\mu\nu} k_\nu$ . If one takes  $C \rightarrow 0$  using the behaviour of  $K_0(x)$  around  $x \sim 0$ , (84), this becomes

$$= \int d^4k d^8\theta (2\pi)^4 \text{Tr}(\bar{d}^4\Phi)_k \text{Tr}\Phi_{-k} \times \left( \frac{g^2}{2} \right) \times \quad (61)$$

$$\left( i\pi^2 \frac{1}{k_-} \int_0^1 dt \frac{1}{t} \log \frac{(1-t)k_\mu^2}{(k_\mu^2 - tk_i^2)} \right). \quad (62)$$

This is  $-\frac{1}{N}$  times the planar contribution (58). Thus we have confirmed the continuity explicitly.

---

<sup>20</sup>Power-counting shows that the divergences of individual terms are at most linear. Furthermore, the linearly divergent parts vanishes because of rotational symmetry. Thus all the terms are at most logarithmically divergent, and the shifting of integration variables should be legitimate.

### A.3 Some useful formulae

#### A.3.1 Feynman integrals in lightcone gauge

In order to evaluate Feynman integrals in the lightcone gauge, we follow the method explained in [55]. We start from the following integral [55],

$$\int dp_+ dp_- d^{2-\epsilon} p \frac{1}{p_-} e^{-i\alpha p_\mu p^\mu - 2i\rho_\mu p^\mu} = -\frac{1}{\rho_-} \pi^{2-\frac{\epsilon}{2}} (i\alpha)^{\frac{\epsilon}{2}-1} \left( e^{i\frac{1}{\alpha}\rho_\mu \rho^\mu} - e^{i\frac{1}{\alpha}\rho_i \rho^i} \right). \quad (63)$$

The above formula can be derived by performing the Wick rotation (which is allowed due to our use of the Mandelstam prescription for factors such as  $\frac{1}{p_-}$ ), and evaluating the integral over the longitudinal space and the transverse space successively. In this expression, the dimensional regularisation is used for the transverse dimensions (i.e. the dimension of the transverse space is formally altered from 2 to  $2 - \epsilon$ ). For our purpose it is not necessary, and hence we will set  $\epsilon = 0$  from now on.

The details regarding the Wick rotation are as follows. The integration contour for  $p_0$  on the real axis from left to right is rotated counter-clockwise by  $\pi/2$ , such that it coincides with the imaginary axis. It is convenient to define  $ip_0^E = p_0$ . After the rotation,  $p_0^E$  is real, and  $\int dp_0$  is replaced by  $i \int dp_0^E$ . The parameter  $\rho_\mu$  can be either real, pure imaginary, complex for all components. The integral is defined if  $\text{Im}\alpha < 0$ . The integration contour over the parameter  $\alpha$  is rotated clockwise on the complex plane by  $\pi/2$  later.

We define the integral  $I$

$$I = \int d^4 p \frac{1}{p_\mu^2 (p+k)_\mu^2} \frac{1}{p_-} e^{ip_\mu C^{\mu\nu} k_\nu}, \quad (64)$$

and also consider  $I_{\text{planar}}$  defined by

$$I_{\text{planar}} = \int d^4 p \frac{1}{p_\mu^2 (p+k)_\mu^2} \frac{1}{p_-}. \quad (65)$$

Both of them are convergent.

By using a Feynman parameter  $t$ ,  $I$  can be rewritten as

$$I = \int d^4 p \int_0^1 dt \frac{1}{((p+tk)_\mu^2 + t(1-t)k_\mu^2)^2} \frac{1}{p_-} e^{ip_\mu C^{\mu\nu} k_\nu} \quad (66)$$

By using a Schwinger parameter  $\alpha$ , this becomes,

$$\begin{aligned} I &= \int d^4 p \int_0^1 dt (-1) \int_0^{+\infty} d\alpha \alpha e^{-i\alpha((p+tk)_\mu^2 + t(1-t)k_\mu^2)} \frac{1}{p_-} e^{ip_\mu C^{\mu\nu} k_\nu} \\ &= (-1) \int d^4 p \int_0^1 dt \int_0^{+\infty} d\alpha \alpha e^{-i\alpha(p_\mu^2 + 2tp_\mu k^\mu + tk_\mu^2)} \frac{1}{p_-} e^{ip_\mu C^{\mu\nu} k_\nu} \\ &= (-1) \int d^4 p \int_0^1 dt \int_0^{+\infty} d\alpha \alpha e^{-i\alpha p_\mu^2} e^{-2ip_\mu(\alpha tk^\mu - \frac{1}{2}C^{\mu\nu} k_\nu)} e^{-i\alpha tk_\mu^2} \frac{1}{p_-}. \end{aligned} \quad (67)$$

We now apply (63) with

$$\rho^\mu = \alpha tk^\mu - \frac{1}{2}C^{\mu\nu} k_\nu = \alpha tk^\mu - \hat{k}^\mu. \quad (68)$$

We use the notation

$$\tilde{k}^\mu = 2\hat{k}^\mu = C^{\mu\nu}k_\nu \quad (69)$$

for brevity. Then,

$$I = (-1) \int_0^1 dt \int_0^{+\infty} d\alpha \alpha \left( -\frac{1}{\rho_-} \pi^2 (i\alpha)^{-1} \left( e^{i\frac{1}{\alpha}\rho_\mu\rho^\mu} - e^{i\frac{1}{\alpha}\rho_i\rho^i} \right) e^{-i\alpha t k_\mu^2} \right). \quad (70)$$

Since the only non-zero components of  $C$  are  $C^{ij}$ , we have

$$\rho_\mu^2 = \alpha^2 t^2 k_\mu^2 + \hat{k}_i^2, \quad \rho_i^2 = \alpha^2 t^2 k_i^2 + \hat{k}_i^2, \quad \rho_- = \alpha t k_- . \quad (71)$$

We note that  $k_i \hat{k}_i = 0$ . Hence, we obtain

$$I = \pi^2 \int_0^1 dt \int_0^{+\infty} d\alpha \alpha \frac{1}{\alpha t k_-} (i\alpha)^{-1} \left( e^{-i\alpha t(1-t)k_\mu^2 + i\frac{1}{\alpha}\hat{k}_i^2} - e^{-i\alpha(tk_\mu^2 - t^2 k_i^2) + i\frac{1}{\alpha}\hat{k}_i^2} \right). \quad (72)$$

We now rotate the integration contour of  $\alpha$  by writing,  $u = i\alpha$ .<sup>21</sup>

$$\begin{aligned} I &= -\pi^2 \int_0^1 dt \int_0^{+\infty} du u u^{-1} \frac{1}{-it u k_-} \left( e^{-ut(1-t)k_\mu^2 - \frac{1}{u}\hat{k}_i^2} - e^{-u(tk_\mu^2 - t^2 k_i^2) - \frac{1}{u}\hat{k}_i^2} \right) \\ &= -i\pi^2 \frac{1}{k_-} \int_0^1 dt \frac{1}{t} \int_0^{+\infty} du \frac{1}{u} \left( e^{-ut(1-t)k_\mu^2 - \frac{1}{4u}\tilde{k}_i^2} - e^{-u(tk_\mu^2 - t^2 k_i^2) - \frac{1}{4u}\tilde{k}_i^2} \right). \end{aligned} \quad (73)$$

Applying the integral representation of the Bessel function, (79), we finally obtain

$$I = -2i\pi^2 \frac{1}{k_-} \int_0^1 dt \frac{1}{t} \left( K_0 \left( \sqrt{t(1-t)k_\mu^2 k_i^2} \right) - K_0 \left( \sqrt{(tk_\mu^2 - t^2 k_i^2)\tilde{k}_i^2} \right) \right). \quad (74)$$

This is an odd function with respect to  $k_\mu$ .

We proceed similarly for the  $I_{\text{planar}}$ ,

$$I_{\text{planar}} = -i\pi^2 \frac{1}{k_-} \int_0^1 dt \frac{1}{t} \int_0^{+\infty} du \frac{1}{u} \left( e^{-ut(1-t)k_\mu^2} - e^{-u(tk_\mu^2 - t^2 k_i^2)} \right). \quad (75)$$

In order to evaluate this, we consider

$$\int_0^{+\infty} du \frac{1}{u} (e^{-Au} - e^{-Bu}). \quad (76)$$

This integral is convergent as the two terms cancel each other when  $u \sim 0$ . By partial integration, we have

$$\begin{aligned} \int_0^{+\infty} du \frac{1}{u} (e^{-Au} - e^{-Bu}) &= \log u (e^{-Au} - e^{-Bu}) \Big|_0^{+\infty} - \int_0^{+\infty} du \log u (-Ae^{-Au} - (-B)e^{-Bu}) \\ &= -\log \frac{A}{B}. \end{aligned} \quad (77)$$

---

<sup>21</sup>We assume that the external momenta  $k_\mu$  are analytically continued appropriately in order to avoid any problem which may occur in the Wick rotation of the contour.

In order to show the last equality, notice that the two integrals in the second expression are separately convergent, and perform the change of variables  $v = Au$  and  $v = Bu$  for them, respectively.

Finally, we have

$$I_{\text{planar}} = i\pi^2 \frac{1}{k_-} \int_0^1 dt \frac{1}{t} \log \frac{(1-t)k_\mu^2}{(k_\mu^2 - tk_i^2)}. \quad (78)$$

The argument of the log function is 1 for  $t \sim 0$ , so that the integral over the Feynman parameter is convergent. This is an odd function with respect to  $k_\mu$ .

### A.3.2 Bessel functions

The modified Bessel functions of the second kind  $K_\nu(x)$  have an integral representation <sup>22</sup>

$$K_\nu(\sqrt{\beta\gamma}) = 2^{\nu-1} \sqrt{\frac{\gamma}{\beta}} \int_0^{+\infty} e^{-\frac{\beta}{4u} - \gamma u} u^{\nu-1} du. \quad (79)$$

Here we assume  $\beta > 0, \gamma > 0$ .

Their behaviours at  $x \sim 0$  are governed, for integer valued  $\nu$ , by the expansion

$$\begin{aligned} K_n(x) &= \frac{1}{2} \left(\frac{x}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(-\frac{x^2}{4}\right)^k + (-1)^{n+1} \log\left(\frac{x}{2}\right) I_n(x) \\ &+ (-1)^n \frac{1}{2} \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{1}{(n+k)!k!} (\psi(k+1) + \psi(n+k+1)) \left(\frac{x^2}{4}\right)^k, \end{aligned} \quad (80)$$

where  $I_\nu(x)$  is defined by

$$I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{+\infty} \frac{\left(\frac{x^2}{4}\right)^n}{n! \Gamma(n+\nu+1)} \quad (81)$$

and  $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ . Specifically, we have

$$K_0(x) = -\log \frac{x}{2} \times I_0(x) + \sum_{k=0}^{+\infty} \frac{1}{(k!)^2} \psi(k+1) \left(\frac{x^2}{4}\right)^k, \quad (82)$$

$$I_0(x) = \sum_{k=0}^{+\infty} \frac{1}{(k!)^2} \left(\frac{x^2}{4}\right)^k. \quad (83)$$

Writing down the first few terms, we have

$$K_0(x) = -\log \frac{x}{2} \times \left(1 + \frac{x^2}{4} + \dots\right) + \left(\psi(1) + \psi(2) \frac{x^2}{4} + \dots\right). \quad (84)$$

---

<sup>22</sup>We follow the convention of [56].

## References

- [1] M. R. Douglas and N. A. Nekrasov, “Noncommutative field theory,” *Rev. Mod. Phys.* **73**, 977 (2001) [arXiv:hep-th/0106048].
- [2] S. Minwalla, M. Van Raamsdonk and N. Seiberg, “Noncommutative perturbative dynamics,” *JHEP* **0002**, 020 (2000) [arXiv:hep-th/9912072].
- [3] J. Goldstone, unpublished (1982).
- [4] J. Hoppe, “Quantum theory of a massless relativistic surface and a two-dimensional bound state problem”, PhD Thesis, MIT 1982 (<http://dspace.mit.edu/handle/1721.1/15717>).
- [5] B. de Wit, J. Hoppe and H. Nicolai, “On the quantum mechanics of supermembranes,” *Nucl. Phys. B* **305**, 545 (1988).
- [6] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, “M theory as a matrix model: A conjecture,” *Phys. Rev. D* **55**, 5112 (1997) [arXiv:hep-th/9610043].
- [7] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, “A Large N reduced model as superstring,” *Nucl. Phys. B* **498** (1997) 467 [hep-th/9612115].
- [8] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” *Adv. Theor. Math. Phys.* **2**, 231 (1998) [*Int. J. Theor. Phys.* **38**, 1113 (1999)] [arXiv:hep-th/9711200].
- [9] S. Catterall, D. Schaich, P. H. Damgaard, T. DeGrand and J. Giedt, “N=4 Supersymmetry on a Space-Time Lattice,” *Phys. Rev. D* **90** (2014) 065013 [arXiv:1405.0644 [hep-lat]].  
S. Catterall, P. H. Damgaard, T. Degrand, R. Galvez and D. Mehta, “Phase Structure of Lattice N=4 Super Yang-Mills,” *JHEP* **1211**, 072 (2012) [arXiv:1209.5285 [hep-lat]].
- [10] M. Hanada, S. Matsuura and F. Sugino, “Two-dimensional lattice for four-dimensional N=4 supersymmetric Yang-Mills,” *Prog. Theor. Phys.* **126** (2011) 597 [arXiv:1004.5513 [hep-lat]].  
M. Hanada, “A proposal of a fine tuning free formulation of 4d N = 4 super Yang-Mills,” *JHEP* **1011** (2010) 112 [arXiv:1009.0901 [hep-lat]].
- [11] J. M. Maldacena, M. M. Sheikh-Jabbari and M. Van Raamsdonk, “Transverse fivebranes in matrix theory,” *JHEP* **0301**, 038 (2003) [arXiv:hep-th/0211139].
- [12] M. Unsal, “Supersymmetric deformations of type IIB matrix model as matrix regularization of N = 4 SYM,” *JHEP* **0604**, 002 (2006) [arXiv:hep-th/0510004].  
B. Ydri, “A Proposal for a Non-Perturbative Regularization of N=2 SUSY 4D Gauge Theory,” *Mod. Phys. Lett. A* **22**, 2565 (2007) [arXiv:0708.3066 [hep-th]].
- [13] T. Ishii, G. Ishiki, S. Shimasaki and A. Tsuchiya, “N=4 Super Yang-Mills from the Plane Wave Matrix Model,” *Phys. Rev. D* **78**, 106001 (2008) [arXiv:0807.2352 [hep-th]].  
G. Ishiki, S. W. Kim, J. Nishimura and A. Tsuchiya, “Deconfinement phase transition in N=4 super Yang-Mills theory on  $R \times S^3$  from supersymmetric matrix quantum mechanics,” *Phys. Rev. Lett.* **102**, 111601 (2009) [arXiv:0810.2884 [hep-th]].



- G. Ishiki, S. W. Kim, J. Nishimura and A. Tsuchiya, “Testing a novel large- $N$  reduction for  $N=4$  super Yang-Mills theory on  $R \times S^3$ ,” *JHEP* **0909**, 029 (2009) [arXiv:0907.1488 [hep-th]].
- M. Honda, G. Ishiki, S. W. Kim, J. Nishimura and A. Tsuchiya, “Direct test of the AdS/CFT correspondence by Monte Carlo studies of  $N=4$  super Yang-Mills theory,” *JHEP* **1311**, 200 (2013) [arXiv:1308.3525 [hep-th]].
- [14] H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, “Noncommutative Yang-Mills in IIB matrix model,” *Nucl. Phys. B* **565**, 176 (2000) [arXiv:hep-th/9908141].
- [15] R. C. Myers, “Dielectric-branes,” *JHEP* **9912**, 022 (1999) [arXiv:hep-th/9910053].
- [16] D. B. Kaplan and M. Unsal, “A Euclidean lattice construction of supersymmetric Yang-Mills theories with sixteen supercharges,” *JHEP* **0509**, 042 (2005) [hep-lat/0503039].
- F. Sugino, “Various super Yang-Mills theories with exact supersymmetry on the lattice,” *JHEP* **0501**, 016 (2005) [hep-lat/0410035].
- S. Catterall, “A Geometrical approach to  $N=2$  super Yang-Mills theory on the two dimensional lattice,” *JHEP* **0411**, 006 (2004) [hep-lat/0410052].
- [17] I. Kanamori and H. Suzuki, “Restoration of supersymmetry on the lattice: Two-dimensional  $N = (2,2)$  supersymmetric Yang-Mills theory,” *Nucl. Phys. B* **811**, 420 (2009) [arXiv:0809.2856 [hep-lat]].
- M. Hanada and I. Kanamori, “Absence of sign problem in two-dimensional  $N = (2,2)$  super Yang-Mills on lattice,” *JHEP* **1101**, 058 (2011) [arXiv:1010.2948 [hep-lat]].
- [18] S. Catterall, D. B. Kaplan and M. Unsal, “Exact lattice supersymmetry,” *Phys. Rept.* **484**, 71 (2009) [arXiv:0903.4881 [hep-lat]].
- [19] N. Seiberg, L. Susskind and N. Toumbas, “Space-time noncommutativity and causality,” *JHEP* **0006**, 044 (2000) [hep-th/0005015].
- [20] L. Alvarez-Gaume and J. L. F. Barbon, “Nonlinear vacuum phenomena in noncommutative QED,” *Int. J. Mod. Phys. A* **16**, 1123 (2001) [hep-th/0006209].
- [21] J. Gomis and T. Mehen, “Space-time noncommutative field theories and unitarity,” *Nucl. Phys. B* **591**, 265 (2000) [hep-th/0005129].
- [22] A. Matusis, L. Susskind and N. Toumbas, “The IR/UV connection in the non-commutative gauge theories,” *JHEP* **0012**, 002 (2000) [arXiv:hep-th/0002075].
- [23] S. Mandelstam, “Light Cone Superspace And The Ultraviolet Finiteness Of The  $N=4$  Model,” *Nucl. Phys. B* **213**, 149 (1983).
- [24] L. Brink, O. Lindgren and B. E. W. Nilsson, “The Ultraviolet Finiteness Of The  $N=4$  Yang-Mills Theory,” *Phys. Lett. B* **123**, 323 (1983).
- [25] S. Ananth, S. Kovacs and H. Shimada, “Proof of all-order finiteness for planar beta-deformed Yang-Mills,” *JHEP* **0701**, 046 (2007) [arXiv:hep-th/0609149].

- [26] S. Ananth, S. Kovacs and H. Shimada, “Proof of ultra-violet finiteness for a planar non-supersymmetric Yang-Mills theory,” Nucl. Phys. B **783**, 227 (2007) [arXiv:hep-th/0702020].
- [27] I. Jack and D. R. T. Jones, “Ultraviolet finite noncommutative theories,” Phys. Lett. B **514** (2001) 401 [hep-th/0105221].
- [28] S. Weinberg, “High-energy behavior in quantum field theory,” Phys. Rev. **118**, 838 (1960).
- [29] A. Smith, “Light Cone Formulation On N=2 Yang-mills,” Nucl. Phys. B **261** (1985) 285.
- [30] M. Hayakawa, “Perturbative analysis on infrared aspects of noncommutative QED on  $R^{*4}$ ,” Phys. Lett. B **478** (2000) 394 [hep-th/9912094].
- [31] M. Hayakawa, “Perturbative analysis on infrared and ultraviolet aspects of noncommutative QED on  $R^{*4}$ ,” hep-th/9912167.
- [32] A. Gonzalez-Arroyo and M. Okawa, “A Twisted Model for Large  $N$  Lattice Gauge Theory,” Phys. Lett. B **120**, 174 (1983).  
 “The Twisted Eguchi-Kawai Model: A Reduced Model for Large  $N$  Lattice Gauge Theory,” Phys. Rev. D **27**, 2397 (1983).
- [33] V. V. Khoze and G. Travaglini, “Wilsonian effective actions and the IR / UV mixing in noncommutative gauge theories,” JHEP **0101** (2001) 026 [hep-th/0011218].
- [34] D. Zanon, “Noncommutative N=1, N=2 super U(N) Yang-Mills: UV / IR mixing and effective action results at one loop,” Phys. Lett. B **502** (2001) 265 [hep-th/0012009].
- [35] A. A. Bichl, M. Ertl, A. Gerhold, J. M. Grimstrup, H. Grosse, L. Popp, V. Putz and M. Schweda *et al.*, “Noncommutative U(1) superYang-Mills theory: Perturbative selfenergy corrections,” Int. J. Mod. Phys. A **19** (2004) 4231 [hep-th/0203141].
- [36] A. Armoni, R. Minasian and S. Theisen, “On Noncommutative N=2 superYang-Mills,” Phys. Lett. B **513** (2001) 406 [hep-th/0102007].
- [37] D. Zanon, “Noncommutative perturbation in superspace,” Phys. Lett. B **504** (2001) 101 [hep-th/0009196].  
 A. Santambrogio and D. Zanon, “One loop four point function in noncommutative N=4 Yang-Mills theory,” JHEP **0101** (2001) 024 [hep-th/0010275].  
 M. Pernici, A. Santambrogio and D. Zanon, “The One loop effective action of noncommutative N=4 superYang-Mills is gauge invariant,” Phys. Lett. B **504** (2001) 131 [hep-th/0011140].
- [38] A. F. Ferrari, H. O. Girotti, M. Gomes, A. Y. Petrov, A. A. Ribeiro, V. O. Rivelles and A. J. da Silva, “Towards a consistent noncommutative supersymmetric Yang-Mills theory: Superfield covariant analysis,” Phys. Rev. D **70** (2004) 085012 [hep-th/0407040].
- [39] E. T. Whittaker and G. N. Watson, “A Course of Modern Analysis,” Cambridge University Press; 4th edition (January 2, 1927), Section 4.44.

- [40] M. T. Grisaru, W. Siegel and M. Rocek, “Improved Methods for Supergraphs,” Nucl. Phys. B **159**, 429 (1979).
- [41] L. Brink, O. Lindgren and B. E. W. Nilsson, “N=4 Yang-Mills Theory On The Light Cone,” Nucl. Phys. B **212**, 401 (1983).
- [42] S. J. Gates, M. T. Grisaru, M. Rocek and W. Siegel, “Superspace Or One Thousand and One Lessons in Supersymmetry,” hep-th/0108200.
- [43] N. Itzhaki, J. M. Maldacena, J. Sonnenschein and S. Yankielowicz, “Supergravity and the large N limit of theories with sixteen supercharges,” Phys. Rev. D **58**, 046004 (1998).
- [44] M. Hanada, Y. Hyakutake, G. Ishiki and J. Nishimura, “Holographic description of a quantum black hole on a computer,” Science **344**, 882 (2014) [arXiv:1311.5607 [hep-th]].
- [45] S. Catterall, A. Joseph and T. Wiseman, “Thermal phases of D1-branes on a circle from lattice super Yang-Mills,” JHEP **1012** (2010) 022 [arXiv:1008.4964 [hep-th]];  
 “Gauge theory duals of black hole - black string transitions of gravitational theories on a circle,” arXiv:1009.0529 [hep-th].
- [46] A. Hashimoto and N. Itzhaki, “Non-commutative Yang-Mills and the AdS/CFT correspondence,” Phys. Lett. B **465**, 142 (1999) [arXiv:hep-th/9907166].
- [47] J. M. Maldacena and J. G. Russo, “Large N limit of non-commutative gauge theories,” JHEP **9909**, 025 (1999) [arXiv:hep-th/9908134].
- [48] T. Matsumoto and K. Yoshida, “Integrability of classical strings dual for noncommutative gauge theories,” JHEP **1406**, 163 (2014) [arXiv:1404.3657 [hep-th]].
- [49] S. Kawamoto, T. Kuroki and D. Tomino, “Renormalization group approach to matrix models via noncommutative space,” JHEP **1208** (2012) 168 [arXiv:1206.0574 [hep-th]].
- [50] S. Vaidya, “Perturbative dynamics on the fuzzy  $S^{*2}$  and  $RP^{*2}$ ,” Phys. Lett. B **512** (2001) 403 [hep-th/0102212].
- [51] C. S. Chu, J. Madore and H. Steinacker, “Scaling limits of the fuzzy sphere at one loop,” JHEP **0108** (2001) 038 [hep-th/0106205].
- [52] H. Steinacker, “Emergent Geometry and Gravity from Matrix Models: an Introduction,” Class. Quant. Grav. **27**, 133001 (2010) [arXiv:1003.4134 [hep-th]].
- [53] P. Aschieri, M. Dimitrijevic, F. Meyer and J. Wess, “Noncommutative geometry and gravity,” Class. Quant. Grav. **23**, 1883 (2006) [hep-th/0510059].
- [54] S. Ananth, S. Kovacs and S. Parikh, “Gauge-invariant correlation functions in light-cone superspace,” JHEP **1205** (2012) 096 [arXiv:1203.5376 [hep-th]].
- [55] D. M. Capper, J. J. Dulwich and M. J. Litvak, “On the Evaluation of Integrals in the Light Cone Gauge,” Nucl. Phys. B **241** (1984) 463.
- [56] NIST Digital Library of Mathematical Functions. <http://dlmf.nist.gov/>, Release 1.0.8 of 2014-04-25.